ABOUT COORDINATES ON THE PHASE-SPACES OF SCHLESINGER SYSTEM ($n+1$ MATRICES, $sl(2,\mathbb{C})$-CASE) AND GARNIER–PAINLEVÉ 6 SYSTEM

MIKHAIL V. BABICH

ABSTRACT. The geometric model of the pathway linking Schlesinger and Garnier–Painlevé 6 systems based on an original orthonormalization of a set of elements in $sl(2,\mathbb{C})$ is constructed. The explicit polynomial map of the Cartesian products of $n-2$ quadrics (the Zariski-topology chart of the phase space of the Garnier–Painlevé 6 system) into the phase space of the Schlesinger system and the rational inverse to this map are presented.

The subject of our considerations will be the phase space of the Schlesinger system (SchS) of equations (about SchS and around it see, for example, [1]):

$$dA^{(k)} = \sum_{i\neq k}[A^{(i)}, A^{(k)}]d\log(\lambda_i - \lambda_k), \quad \sum_k A^{(k)} = 0 \tag{1}$$

for $n+1$ $sl(2,\mathbb{C})$-valued matrices $A^{(k)}$ depending on complex parameters $\lambda_0, \lambda_1, \dots, \lambda_n$.

Let us introduce the notations. We denote $\langle A, B\rangle = tr\ AB$ — the Killing product. It set the isomorphism between $sl(2,\mathbb{C})$ and $sl^*(2,\mathbb{C})$; by the matrix elements of $|A\rangle \in sl^*(2,\mathbb{C})$ we mean the matrix elements of the corresponding $A \in sl(2,\mathbb{C})$.

Denote $a_{ij} := \langle A^{(i)}, A^{(j)}\rangle$, $f_{ijk} := \langle [A^{(i)}, A^{(j)}], A^{(k)}\rangle$. The values $a_{ii} = -2\det A^{(i)}$, $i = 0, 1, \dots, n$ keep constant values for the solutions of (1), they are the parameters of SchS.

Let $SL|\overset{a_{kk}}{A}\rangle$ be the orbit of the co-adjoint action of $SL(2,\mathbb{C})$ on nonzero element $|\overset{a_{kk}}{A}\rangle$ such that $\langle \overset{a_{kk}}{A}, \overset{a_{kk}}{A}\rangle = a_{kk}$. Note that $A^{(k)} \in SL|\overset{a_{kk}}{A}\rangle$ if $A^{(k)} \neq 0$ only, so for zero values of $A^{(k)}$ we need to extend the orbit. We blow up the vertex of cone $\langle \overset{0}{A}, \overset{0}{A}\rangle = 0$.

We define $SL|A\rangle'$, $\langle A, A\rangle = R^2$ as the submanifold of $sl(2,\mathbb{C}) \times \mathbf{P}SL|A\rangle$. Its points have "the affine component" A and "the projective component" $\tilde{A}$:

$$\left(A, \tilde{A}\right) = \left(\begin{pmatrix} X_3 & X_1 \\ X_2 & -X_3 \end{pmatrix}, \begin{pmatrix} x_3 & x_1 \\ x_2 & -x_3 \end{pmatrix}\right) \in SL|A\rangle'.$$

The submanifold is defined by the equations

$$X_1X_2 + X_3^2 = R^2/2, \quad x_1x_2 + x_3^2 = x_0^2R^2/2, \quad X_i = x_0x_i, \quad i = 1, 2, 3.$$

We can see that $SL|A\rangle'$ is an algebraic simplectic manifold with the form induced by the restriction of the Lie-Poisson bracket on the orbit:

$$\omega_{LP}^R = dX_3 \wedge d\log X_1 = -dX_3 \wedge d\log X_2 = d\frac{X_2}{\sqrt{R^2/2} - X_3} \wedge dX_1 = d\frac{X_1}{\sqrt{R^2/2} + X_3} \wedge dX_2\ ;$$

Key words and phrases. Schlesinger system, Painlevé equations, isomonodromic deformations, phase space, simplectic coordinates.

if $R^2=0$, $\omega^0_{LP} = d(x_3:x_1)\wedge dX_1 = d(-x_3:x_2)\wedge dX_2$.

The simplectic manifold $SL|A\rangle'$ can be covered by two standard simplectic charts $(\mathbb{A}^2, dp\wedge dq)$, $(\mathbb{A}^2, dp'\wedge dq')$: with the transition functions $p'=1/p, \quad q'=-q(pq+2\sqrt{R^2/2})$, for example $(p,q) = ((X_3-\sqrt{R^2/2})/X_1, X_1)$, $(p',q') = (X_1/(X_3-\sqrt{R^2/2}), X_2)$.

Traditionally the manifold $SL|A\rangle'$ is defined as the abstract manifold, covered by these two charts, see [2, 3].

Note 1. *The manifold $SL|A\rangle'$ is simplectomorphic to the smooth quadric $X_1X_2+X_3^2 = R^2/2$, $R^2\neq 0$ or to the cone with blowing up vertex; these simplectomorphisms are rational.*

We call $SL|\overset{a_{kk}}{A}\rangle'$ "the quadric" in all cases, there are the quadrics that we talk about in the Abstract, *the phase space of the SchS is the submanifold $\sum A^{(i)}=0$ of the Cartesian product of such quadrics.*

__**Restriction**__

We exclude the special case, the sets all the matrices $A^{(i)}$, $i=0,\ldots,n$ can be carried into the upper-triangle form simultaneously.

This case is much more simple then the general one, but needs another method; let us denote by $\Delta_{a_{\overline{ii}}}$ such "all-triangle" sets $A^{(\bar{i})}$ and put

$$\widetilde{\widetilde{M}}_{a_{\overline{ii}}} := \left(SL|\overset{a_{00}}{A}\rangle' \times SL|\overset{a_{11}}{A}\rangle' \times \cdots \times SL|\overset{a_{nn}}{A}\rangle'\right) \setminus \Delta_{a_{\overline{ii}}}$$

The bar over index means it is the set of such values with all values of indexes: $\widetilde{\widetilde{M}}_{a_{\overline{ii}}} = \widetilde{\widetilde{M}}_{a_{00},a_{11},\ldots,a_{nn}}$; we will use the similar notation – $A^{(\bar{i})}$ is equivalent to $A^{(0)}, A^{(1)}, \ldots, A^{(n)}$ etc.

Manifold $\widetilde{\widetilde{M}}_{a_{\overline{ii}}}$ is $2(n+1)$-dimensional simplectic space with the form $\widetilde{\widetilde{\omega}} := \sum_{k=0}^n \omega^{(k)}$, $\omega^{(k)}$ is the form $\omega_{LP}^{\sqrt{a_{kk}}}$ on the "k"-th Cartesian factor $SL|\overset{a_{kk}}{A}\rangle'$.

The group $SL(2,\mathbb{C})$ acts on $\mathbf{P}SL|A\rangle$ (the projectivization commutes with the co-adjoint action). We define (the component-wise) action of $SL(2,\mathbb{C})$ on $\widetilde{\widetilde{M}}_{a_{\overline{ii}}}$ and denote this action by a sub-index:

$$g\left(A'^{(\bar{i})}\right) := A'^{(\bar{i})}_g := (A_g^{(0)}, \tilde{A}_g^{(0)}), \ldots, (A_g^{(n)}, \tilde{A}_g^{(n)}),\ \ A_g^{(i)} := gA^{(i)}g^{-1},\ \tilde{A}_g^{(i)} := g\tilde{A}^{(i)}g^{-1}.$$

Denote the submanifold of $\widetilde{\widetilde{M}}_{a_{\overline{ii}}}$ that consists of such $A'^{(\bar{i})}$ that $\sum A^{(k)}=0$ by $\widetilde{M}_{a_{\overline{ii}}}$. It is evident that such action of $SL(2,\mathbb{C})$ preserve the property $\sum A^{(k)}=0$, so group $SL(2,\mathbb{C})$ acts on $\widetilde{M}_{a_{\overline{ii}}}$ too.

The phase space $M_{a_{\overline{ii}}}$ of the Garnier system (if $n=3$ it is the Painlevé 6-system) is the quotient of $\widetilde{M}_{a_{\overline{ii}}}$ with respect to this action: $M_{a_{\overline{ii}}} := \widetilde{M}_{a_{\overline{ii}}}/SL(2,\mathbb{C})$, it is the main object of our building.

We need functions (coordinates) on the quotient of with respect to $SL(2,\mathbb{C})$. The method is based on the proposition.

Let $\sigma^{(\bar{i})} = \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}$ be a basis of $sl(2,\mathbb{C})$, let $\sigma_g^{(\bar{i})} = g\sigma^{(1)}g^{-1}, g\sigma^{(2)}g^{-1}, g\sigma^{(3)}g^{-1}$ be this basis "turned" by an element $g\in SL(2,\mathbb{C})$, and $SL\sigma^{(\bar{i})} := \bigcup_{g\in SL(2,\mathbb{C})} \sigma_g^{(\bar{i})}$ be the orbit of the action of $SL(2,\mathbb{C})$ on $\sigma^{(\bar{i})}$.

Let the map $\widetilde{M}_{a_{\overline{ii}}} \to SL\sigma^{(\bar{i})}$: $A'^{(\bar{i})} \to \sigma^{(1)}(A'^{(\bar{i})}), \sigma^{(2)}(A'^{(\bar{i})}), \sigma^{(3)}(A'^{(\bar{i})})$ commutes with the action of $SL(2,\mathbb{C})$:

$$\sigma^{(1)}(A'^{(\bar{i})}_g), \sigma^{(2)}(A'^{(\bar{i})}_g), \sigma^{(3)}(A'^{(\bar{i})}_g) = \sigma^{(1)}_g(A'^{(\bar{i})}), \sigma^{(2)}_g(A'^{(\bar{i})}), \sigma^{(3)}_g(A'^{(\bar{i})}).$$

We call this property *the $SL(2,\mathbb{C})$-invariance.*

Proposition 1. *The coordinates of all $A^{(k)}$ in the basis $\sigma^{(\bar{i})}(A'^{(\bar{i})})$ constructed by an $SL(2,\mathbb{C})$-invariant map do not depend on the action of $SL(2,\mathbb{C})$ on $\widetilde{M}_{a_{\overline{ii}}}$, they are functions on $\widetilde{M}_{a_{\overline{ii}}}/SL(2,\mathbb{C})$.* $\square$

There are the functions we will use as the coordinates on $M_{a_{\overline{ii}}}$.

Note 2. *A related map is the routine orthonormalization of a basis in $\mathbb{R}^n$, but we will use some special method based on the Lie-structure of $sl(2,\mathbb{C})$.*

Consider such bases $\sigma'_{\pm,3}$ that

(2) $\langle\sigma'_-, \sigma'_-\rangle = \langle\sigma'_+, \sigma'_+\rangle = \langle\sigma'_-, \sigma'_3\rangle = \langle\sigma'_+, \sigma'_3\rangle = 0,\ \ \langle\sigma'_+, \sigma'_-\rangle = 1,\ \ [\sigma'_+, \sigma'_-] = \sigma_3.$

Denote the set of these bases $SL\sigma_{\pm,3}$, we call them "*the standard*" bases. The foundation of the offered method is the following proposition.

Proposition 2. *For any $A^{(0)} \neq 0$ and any sign of the square root, the value $\sqrt{2a_{00}} := \sqrt{-4\det A^{(0)}}$ is the eigenvalue of the linear operator $ad_{A^{(0)}} = [A^{(0)}, \cdot] : sl(2,\mathbb{C}) \to sl(2,\mathbb{C})$. The corresponding eigenspace is one-dimensional, isotropic and orthogonal to $A^{(0)}$.*

Proof. All statements are $SL(2,\mathbb{C})$-invariant, consequently we can carry $A^{(0)}$ into the Jordan form. For such matrices the Proposition is evident. $\square$

Denote this eigenspace by $(\sigma^{(\sqrt{2a_{00}})})$, and by $\sigma^{(\sqrt{2a_{00}})}$ any eigenvector from it. We define $(\sigma^{(\sqrt{2a_{ii}})})$ for $A'^{(i)}$ from the blowing up vertices now. If $a_{ii} = 0$ and $A^{(i)} \neq 0$, $(\sigma^{(\sqrt{2a_{ii}})})$ is the direction of $\tilde{A}^{(i)}$, we set $(\sigma^{(\sqrt{2a_{ii}})})$ *the direction of $\tilde{A}^{(i)}$ for $A^{(i)} = 0$ too.*

Consider three vectors $A^{(0)}$, $A^{(n-1)}$, $A^{(n)}$ and fix any values of $\sqrt{2a_{00}}$, $\sqrt{2a_{n-1n-1}}$.

We call the standard basis *accompanying to $A^{(0)}$, $A^{(n-1)}$, $A^{(n)}$ along $(\sigma^{(-\sqrt{2a_{00}})})$ and $(\sigma^{(\sqrt{2a_{n-1n-1}})})$* if

$$\sigma_- \in (\sigma^{(-\sqrt{2a_{00}})}),\quad \sigma_+ \in (\sigma^{(\sqrt{2a_{n-1n-1}})}),\quad \langle\sigma_-, A^{(n)}\rangle = 1. \tag{3}$$

Proposition 3. *The standard basis accompanying to $A'^{(0)}$, $A'^{(n-1)}$, $A'^{(n)}$ along $(\sigma^{(-\sqrt{2a_{00}})})$ and $(\sigma^{(\sqrt{2a_{n-1n-1}})})$ exists and unique if and only if*

$$\textbf{\textit{a)}}\ \ \langle\sigma^{(-\sqrt{2a_{00}})}, \sigma^{(\sqrt{2a_{n-1n-1}})}\rangle \neq 0\ ,\ \textit{and}\qquad \textbf{\textit{b)}}\ \ \langle\sigma^{(-\sqrt{2a_{00}})}, A^{(n)}\rangle \neq 0. \tag{4}$$

Proof. If "**a)**" and "**b)**" are satisfied, the explicit formulae

$$\begin{aligned}
\sigma_- &= \sigma^{(-\sqrt{2a_{00}})}\frac{1}{\langle\sigma^{(-\sqrt{2a_{00}})}, A^{(n)}\rangle}\\
\sigma_+ &= \sigma^{(\sqrt{2a_{n-1n-1}})}\frac{\langle\sigma^{(-\sqrt{2a_{00}})}, A^{(n)}\rangle}{\langle\sigma^{(-\sqrt{2a_{00}})}, \sigma^{(\sqrt{2a_{n-1n-1}})}\rangle}\\
\sigma_3 &= [\sigma^{(\sqrt{2a_{n-1n-1}})}, \sigma^{(-\sqrt{2a_{00}})}]\frac{1}{\langle\sigma^{(-\sqrt{2a_{00}})}, \sigma^{(\sqrt{2a_{n-1n-1}})}\rangle}
\end{aligned} \tag{5}$$

give the desired basis. It is unique.

If "**a)**" does not satisfied, vectors σ_- and σ_+ are linear dependent, $\sigma_{\pm,3}$ is not a basis. If "**b)**" does not satisfied, (3) can not take place. $\Box$

The subset of $\widetilde{M}_{a_{\overline{ii}}}$ where (4) fulfilled we denote by $\widetilde{\mathcal{U}}\,(-\sqrt{2a_{00}}//\sqrt{2a_{n-1n-1}}/n)$, or $\widetilde{\mathcal{U}}\,(\,\dots)$ for short:

$$\widetilde{\mathcal{U}}\,(\,\dots) := \{A'^{(\overline{i})} \in \widetilde{M}_{a_{\overline{ii}}} : \quad \langle \sigma^{(-\sqrt{2a_{00}})}, \sigma^{(\sqrt{2a_{n-1n-1}})}\rangle \neq 0, \quad \langle \sigma^{(-\sqrt{2a_{00}})}, A^{(n)}\rangle \neq 0\}$$

From the Restriction follows

Theorem 1. *For any preassigned values of square roots* $\sqrt{2a_{ii}}, \quad i \in \{0, 1, \dots, n\}$

$$\widetilde{M}_{a_{\overline{ii}}} = \bigcup_{\substack{i,j \\ 0\neq i\neq j\neq 0}} \widetilde{\mathcal{U}}\,(-\sqrt{2a_{00}}//\sqrt{2a_{ii}}/j)$$

$\Box$

The Restriction implies that the values $a_{ik},\ f_{ijk}$ characterize the set $A^{(\overline{i})}$ up to the action of $SL(2,\mathbb{C})$, consequently the quotient $M_{a_{\overline{ii}}} := \widetilde{M}_{a_{\overline{ii}}}\ /SL(2,\mathbb{C})$ *is the manifold* . If $a_{kk} \neq 0$ it can be embedded into the space of all $a_{ij},\ f_{ijk}$; for the sets with $A'^{(k_0)}$ on the blowing up divisor some coordinates should be taken from the conic $\langle \tilde{A}^{(k_0)}, \tilde{A}^{(k_0)}\rangle = 0$ on the plane $\mathbb{C}\mathbf{P}^2 \ni \langle \tilde{A}^{(k_0)}, A^{(i_1)}\rangle : \langle \tilde{A}^{(k_0)}, A^{(i_2)}\rangle : \langle \tilde{A}^{(k_0)}, [A^{(i_1)}, A^{(i_2)}]\rangle$.

We treat $M_{a_{\overline{ii}}}$ as the abstract manifold. The quotation of $\widetilde{M}_{a_{\overline{ii}}}$ with respect to $SL(2,\mathbb{C})$ we denote by $\tilde{\pi}$:

$$\tilde{\pi}: \ \widetilde{M}_{a_{\overline{ii}}} \longrightarrow M_{a_{\overline{ii}}} := \widetilde{M}_{a_{\overline{ii}}}\ /SL(2,\mathbb{C}).$$

Denote the embedding of $M_{a_{\overline{ii}}}$ into the space of values $a_{ij},\ f_{ijk}$ and ratios $\langle A^{(i)}, \tilde{A}^{(k_0)}\rangle : \langle [A^{(i)}, A^{(j)}], \tilde{A}^{(k_0)}\rangle$, by $M^f_{a_{\overline{ii}}}$.

Consider an another version of the quotation now. Formulae (5) give the map $A^{(\overline{k})} \longrightarrow \sigma_{\pm,3}(A^{(\overline{k})})$ and basis that we talk about in the Proposition 1. The map is defined on $\widetilde{\mathcal{U}}\ (-\sqrt{2a_{00}}//\sqrt{2a_{n-1n-1}}/n) \subset \widetilde{M}_{a_{\overline{ii}}}$ and induce the embedding of $\widetilde{\mathcal{U}}\,(-\sqrt{2a_{00}}//\sqrt{2a_{n-1n-1}}/n)/SL(2,\mathbb{C}) \subset M_{a_{\overline{ii}}}$ into $\approx{M}_{a_{\overline{ii}}}$:

$$(6) \qquad \begin{array}{ll} A^{(0)} = \begin{pmatrix} \sqrt{\frac{a_{00}}{2}} & 0 \\ q_0' & -\sqrt{\frac{a_{00}}{2}} \end{pmatrix} & A^{(i)} = \begin{pmatrix} \beta_i & q_i \\ q_i' & -\beta_i \end{pmatrix} \\ A^{(n-1)} = \begin{pmatrix} \sqrt{\frac{a_{n-1n-1}}{2}} & q_{n-1} \\ 0 & -\sqrt{\frac{a_{n-1n-1}}{2}} \end{pmatrix} & A^{(n)} = \begin{pmatrix} \beta_n & 1 \\ q_n' & -\beta_n \end{pmatrix} \end{array}$$

– it is the form the matrices from the set $A^{(\overline{i})}$ have in the accompanying basis $\sigma_{\pm,3}$. This map is the embedding because in the fixed basis every vector (matrix) is uniquely defined by its coordinates (matrix elements).

Let us reject $A^{(0)}, A^{(n-1)}, A^{(n)}$ from the set $A^{(\overline{i})}$, it is the projection to the Cartesian product of all $SL|\overset{a_{ii}}{A}\rangle'$ except those with $i = 0, n-1, n$

$$\approx{M}_{a_{\overline{ii}}\setminus\{0n-1n\}} := \quad SL|\overset{a_{11}}{A}\rangle' \times \cdots \times SL|\overset{a_{n-2n-2}}{A}\rangle'\ ,$$

it is the mentioned in the Abstract product of $n-2$ quadrics.

This projection is a bijection on the image, the rejected terms can be restored. Denote

$$\beta_\Sigma := \sum_{i=1}^{n-2} \beta_i, \quad q_\Sigma := \sum_{i=1}^{n-2} q_i, \quad q'_\Sigma := \sum_{i=1}^{n-2} q'_i \ .$$

Because of $\sum A^{(i)} = 0$ and $\det A^{(n)} = -a_{nn}/2$

(7)

$$\begin{aligned}
&-q_{n-1} = q_\Sigma + 1,\ q'_{n-1} := 0,\ \beta_{n-1} := \sqrt{a_{n-1n-1}/2},\\
&-\beta_n = \sqrt{a_{00}/2} + \sqrt{a_{n-1n-1}/2} + \beta_\Sigma,\ q'_n = -(\beta_\Sigma + \sqrt{a_{00}/2} + \sqrt{a_{n-1n-1}/2})^2 + a_{nn}/2,\\
&-q'_0 = q'_\Sigma - (\beta_\Sigma + \sqrt{a_{00}/2} + \sqrt{a_{n-1n-1}/2})^2 + a_{nn}/2, \quad q_0 := 0, \quad \beta_0 := \sqrt{a_{00}/2}\ .
\end{aligned}$$

Denote the composition of the embedding (6) and the rejection of $A^{(0)}, A^{(n-1)}, A^{(n)}$ by $\pi_{\{0n-1n\}}$:

$$\pi_{\{0n-1n\}} :\ \widetilde{\mathcal{U}}\ (-\sqrt{2a_{00}}//\sqrt{2a_{n-1n-1}}/n)/SL(2,\mathbb{C}) \longrightarrow \stackrel{\approx}{M}_{a_{\overline{ii}}\setminus\{0n-1n\}}$$

Proposition 4. *Map $\pi_{\{0n-1n\}}$ is the bijection.* $\square$

The space $\stackrel{\approx}{M}_{a_{\overline{ii}}\setminus\{0n-1n\}}$ is the simplectic manifold, as any product of simplectic spaces. Denote its form by $\stackrel{\approx}{\omega}_{\{0n-1n\}}$.

Theorem 2. *The form $\stackrel{\approx}{\omega}|_{\widetilde{M}\subset\stackrel{\approx}{M}}$, that is the restriction of the simplectic form $\stackrel{\approx}{\omega}$ on $\widetilde{\mathcal{U}}\ (-\sqrt{2a_{00}}//\sqrt{2a_{n-1n-1}}/n) \subset\widetilde{M}_{a_{\overline{ii}}}\subset\stackrel{\approx}{M}_{a_{\overline{ii}}}$, coincides with the lifting of $\stackrel{\approx}{\omega}_{\{0n-1n\}}$ on $\widetilde{\mathcal{U}}\ (\ \dots)$: $\tilde{\pi}^* \circ \pi^*_{\{0n-1n\}}\ \stackrel{\approx}{\omega}_{\{0n-1n\}} = \stackrel{\approx}{\omega}|_{\widetilde{M}\subset\stackrel{\approx}{M}}$.*

Proof. The restriction of $\stackrel{\approx}{\omega}$ on $\widetilde{M}_{a_{\overline{ii}}}$ does not depend on the choice of basis of $sl(2,\mathbb{C})$, so we can calculate the sum $\sum_{i=0}^n \omega^{(i)}$ in the accompanying along $(\sigma^{(-\sqrt{2a_{00}})})$ and $(\sigma^{\sqrt{(2a_{n-1n-1})}})$ basis that exists for $A'^{(\bar{i})} \in\widetilde{\mathcal{U}}\ (-\sqrt{2a_{00}}//\sqrt{2a_{n-1n-1}}/n)$.

The point is, in this basis $\omega^{(0)} = \omega^{(n-1)} = \omega^{(n)} = 0$. It is true because $A'^{(0)}$, $A'^{(n-1)}$, $A'^{(n)}$ belong to *the one-dimensional* submanifolds of $SL|\stackrel{a_{ii}}{A}\rangle'$ – to the intersections of quadrics $SL|\stackrel{a_{ii}}{A}\rangle$, $(i = 0, n-1, n)$ and planes $A^{(0)}_{12} = 0$, $A^{(n-1)}_{21} = 0$, $A^{(n)}_{12} = 1$, consequently $\sum_{i=0}^n \omega^{(i)} = \sum_{i=1}^{n-2} \omega^{(i)}$. $\square$

Denote $\mathcal{U}(\ \dots) := \tilde{\pi}(\widetilde{\mathcal{U}}\ (\ \dots)) \subset M_{a_{\overline{ii}}}$. The definition of $\widetilde{\mathcal{U}}\ (\ \dots)$ is $SL(2,\mathbb{C})$-invariant, consequently $\widetilde{\mathcal{U}}\ (\ \dots) = \tilde{\pi}^{-1}(\mathcal{U}(\ \dots))$,

$$M_{a_{\overline{ii}}} = \bigcup_{\substack{i,j\\ 0\neq i\neq j\neq 0}} \mathcal{U}(-\sqrt{2a_{00}}//\sqrt{2a_{ii}}/j)\ ,$$

and manifold $M_{a_{\overline{ii}}}$ *is simplectic manifold.* There is the global simplectic form ω can be glued from the forms on $\mathcal{U}(\ \dots) \simeq\stackrel{\approx}{M}_{a_{\overline{ii}}\setminus\{0ij\}}$

Note 3. *The set of local simplectic coordinates on $M_{a_{\overline{ii}}}$ is a set $(p_i, q_i)_{i=1}^{n-2}$, where (p_i, q_i) is any pair of local simplectic coordinates on the quadric $SL|\stackrel{a_{ii}}{A}\rangle'$.*

Theorem 3. *Coordinate functions a_{ik}, f_{ijk} and the functions β_i, q_i, q'_i, the matrix elements of $A^{(i)}$ in the accompanying basis, are birationally connected.*

Proof. In one direction it is trivial, using the representation (6) and formulae (7) we calculate $tr\ A^{(i)}A^{(j)} = a_{ij}$ and $tr\ [A^{(i)}, A^{(j)}]A^{(k)} = f_{ijk}$. They will be some polynomials of matrix elements $\beta_i, q_i, q'_i,\ \ i = 1, \dots, n-2$.

Consider the opposite direction. The foundation of the construction is the following proposition that can be verified by direct calculation:

Proposition 5. *For any $A, B \in sl(2, \mathbb{C})$ vector $\sigma = \sigma^{(\sqrt{2\langle A,A\rangle})}(B \setminus A)$:*

$$\sigma^{(\sqrt{2\langle A,A\rangle})}(B \setminus A) := \langle A, A\rangle B - \langle A, B\rangle A + \sqrt{\langle A, A\rangle/2}[A, B] \tag{8}$$

satisfy the equality $[A, \sigma] = \sqrt{2\langle A, A\rangle}\sigma$. $\square$

The Restriction guarantee for any set $A^{(\bar{i})}$ there are such $A^{\widehat{0}}$ and $A^{\widehat{n-1}}$ that $\sigma^{(-\sqrt{2a_{00}})}(A^{(\widehat{0})} \setminus A^{(0)}) \neq 0$, $\sigma^{(\sqrt{2a_{n-1n-1}})}(A^{(\widehat{n-1})} \setminus A^{(n-1)}) \neq 0$. We set

(9)

$$\sigma^{(-\sqrt{2a_{00}})} = \sigma^{(-\sqrt{2a_{00}})}(A^{(\widehat{0})} \setminus A^{(0)}), \quad \sigma^{(\sqrt{2a_{n-1n-1}})} := \sigma^{(\sqrt{2a_{n-1n-1}})}(A^{(\widehat{n-1})} \setminus A^{(n-1)}).$$

For $A^{(\bar{i})} \in \widetilde{\mathcal{U}}\ (-\sqrt{2a_{00}}//\sqrt{2a_{n-1n-1}}/n)$ vectors $\sigma^{(-\sqrt{2a_{00}})}$ and $\sigma^{(\sqrt{2a_{n-1n-1}})}$ are linear independent and (5) for (9) give the standard basis $\sigma_{\pm,3}$ accompanying $A^{(\bar{i})}$. The matrix elements β_i, q_i, q'_i are values $\langle\sigma_3/2, A^{(i)}\rangle$, $\langle\sigma_-, A^{(i)}\rangle$, $\langle\sigma_+, A^{(i)}\rangle$, we get the rational representation of them via a_{ij} and f_{ijk}. $\square$

Conclusion.

The SchS is the Hamiltonian system on $\widetilde{M}_{a_{\overline{ii}}} \times \mathbb{C}^{n+1}$:

$$\widetilde{\widetilde{\omega}}\ |_{\widetilde{M}_{a_{\overline{ii}}}} - \sum da_{ij} \wedge d\log(\lambda_i - \lambda_j) = 0, \quad \text{where}$$

$\widetilde{M}_{a_{\overline{ii}}}$ is the submanifold $\sum A^{(i)} = 0$ of $\widetilde{\widetilde{M}}_{a_{\overline{ii}}}$, of the Cartesian product of $n+1$ quadrics, orbits $SL|\overset{a_{kk}}{A}\rangle'$.

The Garnier–Painlevé 6 system is a Hamiltonian system on $(\widetilde{M}_{a_{\overline{ii}}}/SL(2, \mathbb{C})) \times (\overline{\mathbb{C}}^{n+1}/SL(2, \mathbb{C}))$ with the same Hamiltonians:

$$\omega - \sum da_{ij} \wedge d\log(\lambda(t_i) - \lambda(t_j)) = 0.$$

If $n = 3$ we can set $t_1 = 0, t_2 = 1, t_3 = \infty,\ t_0 := t$ and the extended phase space is $M_{a_{\overline{ii}}} \times \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$.

The goal of this paper is to present the new geometrical model that makes visible *why*(how) $M_{a_{\overline{ii}}} := \widetilde{M}_{a_{\overline{ii}}}/SL(2, \mathbb{C})$ is *birationally* simplectomorphic to the Cartesian product of $n-2$ quadrics $SL|\overset{a_{kk}}{A}\rangle'$ (to one quadric in Painlevé 6-case), and not simplectomorphic. The manifold $M_{a_{\overline{ii}}}$ may be covered by *several* neighborhoods, each of them is simplectomorphic to such a product. In the case $n = 3$ (Painlevé 6) there are three neighborhoods (quadrics). If we add to $M_{a_{\overline{ii}}}$ one neighborhood more, new points of which correspond to the solutions of SchS becoming infinity in the moment t, we get the so named *Okamoto surface*, see [2, 3, 4].

In this paper we constructed the special coordinate atlas on $M_{a_{\overline{ii}}}$, each chart $\mathcal{U}(\ \dots)$ of which is isomorphic (in the Zariski topology) to $\widetilde{\widetilde{M}}_{a_{\overline{ii}} \setminus \{0n-1n\}}$, to the Cartesian product of $n-2$ quadrics. The rational simplectic map $\widetilde{M}_{a_{\overline{ii}}} \to \widetilde{\widetilde{M}}_{a_{\overline{ii}} \setminus \{0n-1n\}}$:

$$A^{(\bar{i})} \xrightarrow{(9)\to(8)\to(5)} (\beta_i, q_i, q'_i)_{i=1}^{n-2}$$

where $\beta_i = tr\ A^{(i)}\sigma_3/2$, $q_i = tr\ A^{(i)}\sigma_-$, $q'_i = tr\ A^{(i)}\sigma_+$ and the short arrows mean the substitutions of the corresponding formulae is presented.

The inverse map does not exist because $\tilde{\pi}$ is not injective, it can be considered as the projection of the bundle $\tilde{\pi} : \widetilde{M}_{a_{\overline{ii}}} \to M_{a_{\overline{ii}}}$, its fibre is isomorphic to $SL(2,\mathbb{C})$. We construct *the local section* of this bundle, the rational (polynomial) map $M_{a_{\overline{ii}}} \supset \overset{\approx}{M}_{a_{\overline{ii}}\setminus\{0n-1n\}} \longrightarrow \widetilde{\mathcal{U}}\ (\ \dots) \subset \widetilde{M}_{a_{\overline{ii}}}$ that parameterize the fibres (formula (7)).

This gives the polynomial expression for the Hamiltonian's $a_{ij} = tr\ A^{(i)}A^{(j)}$ in terms of any canonical coordinates $(p_i, q_i)_{i=1}^{n-2}$ on $\overset{\approx}{M}_{a_{\overline{ii}}\setminus\{0n-1n\}} \subset M_{a_{\overline{ii}}}$, it is the pass between SchS and the Garnier–Painlevé 6 systems announced in the Abstract.

The properties (2) are evidently connected with orthonormality; so the $SL(2,\mathbb{C})$-invariant procedure (9) $\to$ (8) $\to$ (5) is *the version of the orthonormalization of the set* $A^{(\bar{i})}$ *of elements of* $sl(2,\mathbb{C})$.

References

[1] K. Iwasaki, H. Kimura, S. Shimomura, M. Yoshida, "From Gauss to Painlevé: a modern theory of special functions," Braunschweig: Vieweg, 1991. (Aspects of mathematics: E; Vol. 16).

[2] K. Okamoto, "Sur les feuilletages associés aux équations du second ordre á points criticues fixes de P. Painlevé, Espaces des conditions initiales," Japan. J. Math., **5**, 1–79, 1979.

[3] T. Shioda, K. Takano, "On some Hamiltonian structures of Painlevé Painlevé systems I," Funkcial. Ekvac., **40**, 271–291, 1997.

[4] M. V. Babich, "Uravnenija Schlesingera i uravnenija Garnier–Painleve 6, geometrija perehoda", PDMI Preprint-06/2006, St. Petersburg, 1–75, 2006. (in Russian), http://www.pdmi.ras.ru/preprint/2006/index.html

PDMI, Fontanka 27, St.Petersburg, 191023, Russia

E-mail address: `mbabich@pdmi.ras.ru`